$e^{\pi i} + 1 = 0$, *what does this mean?*

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Euler's formula, $e^{\pi i} + 1 = 0$, is regarded as the most beautiful equation in mathematics. However, at first sight is just a meaningless heap of symbols. The numbers π and e are mysterious enough. But raising something to the power $i = \sqrt{-1}$ must be a joke. Well, it isn't. The usual strategy of extending the domain of a function gently leads to this equation.

The essence of being a circle: π

The number π is irrational, meaning that it cannot be written as a ratio of two whole numbers. We can only approximate its value.

$$\pi = 3.14159265358979323846264338327950288419 71\ldots$$

To figure out its digits, we can take any circular physical object, measure its diameter and circumference, then π is just their ratio.

$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

Physical measurements are not accurate enough to calculate π with precision, so we need to use some formula. However, π is not just irrational but *transcendental* as well, meaning that it can't be produced by a finite algebraic expression using addition, subtraction, multiplication, division, and rational powers. We need use some infinite sum formula to find π's value.

$\sqrt{2}$ is a solution of $x^2 = 2$, so it is not transcendental.

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Here π is not expressed directly. If the right hand side is evaluated, we can get π's value by a simple arithmetic calculation.

It might be a bit surprising that infinitely many summands add up to a finite value. The trick is that later terms are getting smaller and smaller and contribute less and less to the total. In practice, we can stop after just summing finitely many terms, when we think the approximation is good enough.

The sigma sum notation excels in capturing the pattern of infinite sums.

The base of the natural logarithm: e

The first few digits of e are

$$e = 2.71828182845904523536028747135266249775724709369995\ldots,$$

but these only give finite approximations, since e is also irrational and transcendental. The following limits can be used to calculate the digits. They are converging slowly (big n and small x are needed). The first one comes from compound interest calculation with increasingly frequent compounding.

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$e = \lim_{x\to 0} (1 + x)^{\frac{1}{x}}$$

Probably the easiest way calculate e is to take a couple of terms from the infinite sum

$$e^1 = e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots ,$$

which is just a special case of e^x, by applying the power series form of the exponential function to 1.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Note, that this is a radically different way of calculating the powers of e. Instead of directly taking powers of e we take infinitely many powers of the exponent (or in practice just finitely many of them).

The notation $n!$ stands for n factorial, which is the product of the natural numbers from 1 to n. For zero the value of 0! is 1 by definition. Recursively, $0! = 1$ and $n! = n(n-1)!$.

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2 = 1 \cdot 2 \\ 3! &= 6 = 1 \cdot 2 \cdot 3 \\ 4! &= 24 = 1 \cdot 2 \cdot 3 \cdot 4 \\ 5! &= 120 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \\ 6! &= 720 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \\ 7! &= 5040,\ 8! = 40320, \end{aligned}$$

Why $0! = 1$? To complete the pattern of $(n-1)! = \frac{n!}{n}$. Or, $n!$ is the number of ways to arrange n things, and we can do it in one way when we have zero objects.

Trigonometric functions redefined

Trigonometric functions are defined by the ratios of the sides of right angled triangles. We can extend them to accept bigger angles by using reference angles, which boils down to the coordinates of a point tracing a circular motion. Yet, there is a third way to define trigonometric functions. We can use power series, just as for the exponential function above.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Why do these infinite sums work for these functions? Answer is in Calculus: Taylor and Maclaurin series. Just to get an intuitive idea, one may use a graphical computer algebra system to draw the first few term of the power series.

What is the point? Why calculating the sine and cosine functions in such a roundabout way? Because **power series work for complex numbers** as well, not only for real numbers. I can't make sense of a triangle with a complex angle, but I can easily calculate the nth power of that complex number. They are also useful for numerical calculations carried out by computers.

The surprise: the exponential breaks into two trigonometric halves

It seems now that we can calculate powers like e^{a+bi}, where $a + bi$ is a complex number in the standard form, $a, b \in \mathbb{R}$. By using the simple laws of exponents we have $e^{a+bi} = e^a e^{bi}$. The first factor of the product is e^a is a real number itself, thus we can concentrate on the more interesting e^{bi} part.

We would like to know what it means to have a complex power of e, what is e^{ix} where $x \in \mathbb{R}$.

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \frac{(ix)^0}{0!} + \frac{(ix)^1}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$

By laws of exponents $(ix)^n = i^n x^n$, thus I can separate the real and complex powers.

It is easy to calculate to powers of the complex unit

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1, \end{aligned}$$

to see that they cycle with a period of 4. This gives the repeating patterns $++--$ and real-complex-real-complex.

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \frac{1}{0!} + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} - \cdots$$

We see that i remains in every second term, so let's try to separate those and extract i.

$$\begin{aligned} e^{ix} = &\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right) \\ &+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) \end{aligned}$$

Looks familiar? Sure, it's on the previous page. The sums in the parentheses are trigonometric functions. Therefore, we have Euler's formula, establishing connection between the exponential and trigonometric functions.

By letting $a = \cos(x)$ and $b = \sin(x)$ we are also back to the familiar algebraic form of complex numbers: $a + bi$.

$$e^{ix} = \cos(x) + i\sin(x)$$

In a sense, when we try to break the complex exponential function into two parts, we get the complex sine and cosine. Note, that we only need to evaluate the trigonometric functions on the real value x.

Now we can answer the main question. It is just applying Euler's formula to π:

$$e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1 + i \cdot 0 = -1,$$

therefore,

$$e^{\pi i} + 1 = 0.$$