Breaking the rule - the imaginary unit

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We learned that whenever a negative number appears inside the square root in the quadratic formula, we can just say there are no real number solutions and walk away. Is it really the end of the story? Can we construct something that somehow makes sense logically and looks like a solution of an 'unsolvable' quadratic.

"Mathematics often develops by mathematicians feeling frustrated about being unable to do something in the existing world, so they invent a new world in which they can do it. I like to think of us as inveterate rule-breakers. As soon as we're presented with a rule saying we're not allowed to so something, we want to see if we can make a world in which we can do it. This is very different from the popular conception of mathematics as a subject in which you have to follow a whole load of rules." [1]

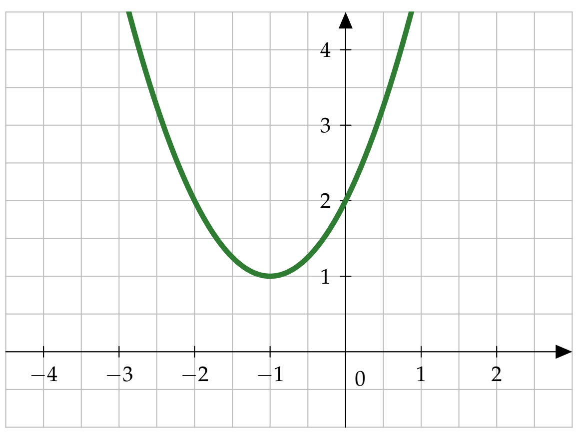
[1] Eugenia Cheng. *Beyond Infinity: An Expedition to the Outer-Limits of the Mathematical Universe*. Profile Books Limited, 2017

Here is a quadratic equation.

$$x^2 + 2x + 2 = 0$$

Its graph shows that the curve never crosses the x axis, thus the equation has no real solutions.

The story given here is not the one that happened historically. Imaginary numbers appeared in intermediate steps for finding roots of cubic equations. It took a couple of generations to get used to the idea, so a bit of mental resistance is normal and expected.



The well-known formula is an automated way of finding solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now $a = 1$, $b = 2$ and $c = 2$.

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

We arrived at the problematic part. The square root of a negative number doesn't make sense, since any real number multiplied by itself will give a non-negative number. There is just no such real number giving a negative square.

Can we do something? What would be the smallest change needed to make this work? Let's try to isolate the core of the problem.

$$\sqrt{-4} = \sqrt{4 \cdot (-1)} = \sqrt{4} \cdot \sqrt{-1} = 2\sqrt{-1}$$

Thus, $\sqrt{-1}$ seems to be the source of the trouble. A useful way to get rid of a problem in mathematics is to hide it behind a symbol. It's tempting to say that $i = \sqrt{-1}$, but that is not exactly right.

Using the rule for real numbers $\sqrt{ab} = \sqrt{a}\sqrt{b}$

The full name of i is the imaginary unit. How appropriate was to call a new kind of number imaginary – well, opinions vary.

$$1 = \sqrt{1} = \sqrt{(-1) \cdot (-1)} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = i^2 = -1$$

Our careless definition made the negative numbers disappear, so we need to be a bit more diplomatic.

$$\sqrt{-a} = i\sqrt{a}, \text{ where } i^2 = -1$$

With this, we can get back to our quadratic equation.

$$x = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm i\sqrt{4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

The answers are $(-1 - i)$ and $(-1 + i)$, but are they really solutions? We can easily check that by substituting them back into the original equation.

$$(-1 + i)^2 + 2(-1 + i) + 2 =?$$

How can we calculate this value? We are used to working with letters in algebraic expressions, so we can just carry on as normal.

$$(-1 + i)^2 + 2(-1 + i) + 2 = 1 - 2i + i^2 - 2 + 2i + 2 = 1 + i^2$$

We didn't really get zero, or did we? We have the rule $i^2 = -1$, as we have just made it up.

$$1 + i^2 = 1 - 1 = 0$$

Right, it is really a solution. The choice $x = -1 + i$ really makes $x^2 + 2x + 2 = 0$ into a true statement.

Same substitution would reveal that $-1 - i$ is also a solution.

What is i then? It is a new kind of mathematical thing we have just invented with the purpose of solving quadratic equations that were unsolvable by previous rules. After adding i as a new kind of number, we can continue our algebraic calculations as before. It will interact with the familiar real numbers by the algebraic operations. The only thing to keep in mind that $i^2 = -1$. **Now there is a whole new world out there to explore!**

Maybe it is better to ask 'What is i good for?'. That question has clear and overwhelming answers in Physics, for example.